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Splitting families and forcing

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Abstract

According to [M.S. Kurilić, Cohen-stable families of subsets of the integers, J. Symbolic Logic 66 (1) (2001) 257–270], adding a Cohen real destroys a splitting family $\mathcal{S}$ on ω if and only if $\mathcal{S}$ is isomorphic to a splitting family on the set of rationals, $\mathbb{Q}$, whose elements have nowhere dense boundaries. Consequently, $|\mathcal{S}| < \mathrm{cov}(\mathcal{M})$ implies the Cohen-indestructibility of $\mathcal{S}$. Using the methods developed in [J. Brendle, S. Yatabe, Forcing indestructibility of MAD families, Ann. Pure Appl. Logic 132 (2–3) (2005) 271–312] the stability of splitting families in several forcing extensions is characterized in a similar way (roughly speaking, destructible families have members with 'small generalized boundaries' in the space of the reals). Also, it is proved that a splitting family is preserved by the Sacks (respectively: Miller, Laver) forcing if and only if it is preserved by some forcing which adds a new (respectively: an unbounded, a dominating) real. The corresponding hierarchy of splitting families is investigated.

Keywords: Splitting families; Forcing; Cohen forcing; Random forcing; Sacks forcing; Miller forcing; Laver forcing; Hechler forcing

1. Introduction

Let ω be the set of natural numbers and $[\omega]^\omega$ the family of its infinite subsets. We will say that the set $S \in [\omega]^\omega$ *splits* the set $X \in [\omega]^\omega$ if and only if $X \cap S$ and $X \setminus S$ are infinite sets. A family $\mathcal{S} \subset [\omega]^\omega$ is called a *splitting family on* ω if and only if each set $X \in [\omega]^\omega$ is split by some element S of $\mathcal{S}$. It is well-known that the 'small cardinal' $\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S}$ is a splitting family on $\omega\}$ is uncountable, so $\aleph_0 < \mathfrak{s} \leq \mathfrak{c}$.

If $\mathbb{P}$ is a forcing notion, a splitting family $\mathcal{S}$ on ω will be called $\mathbb{P}$-*stable* iff $\mathcal{S}$ remains a splitting family in each generic extension (of the ground model) by $\mathbb{P}$. A consequence of Theorem 7 of [8] is the following characterization of Cohen-stable splitting families.

Theorem 1. *Let $\mathbb{R}$ be the real line, $\mathbb{Q}$ the space of rationals and $\mathbb{C}$ the Cohen forcing. If $\mathcal{S}$ is a splitting family on ω, then the following conditions are equivalent:*

(a) *$\mathcal{S}$ is $\mathbb{C}$-stable;*

(b) *For each bijection $f : \mathbb{Q} \to \omega$ there exists $S \in \mathcal{S}$ such that, in the space $\mathbb{R}$, the set $\overline{f^{-1}[S]} \cap \overline{f^{-1}[\omega \setminus S]}$ is not nowhere dense;*

(c) *For each bijection $f : \mathbb{Q} \to \omega$ there exists $S \in \mathcal{S}$ such that, in the space $\mathbb{Q}$, the boundary $\partial f^{-1}[S]$ is not nowhere dense.*

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The corresponding 'forcing-free' characterization of $\mathbb{C}$-stable maximal almost disjoint (mad) families (Theorem 3 of [8], obtained independently by Hrušák in [4]) was generalized for several real forcings by Brendle and Yatabe in [2] and for mad families on uncountable cardinals in [9]. In the present paper, using the technique developed in [2], we generalize Theorem 1 for several forcing notions and investigate forcing stability of splitting families.

In particular we will consider six well-known forcing notions: Sacks, Miller, Laver, Cohen, Solovay and Hechler forcing, denoted by $\mathbb{S}$, $\mathbb{M}$, $\mathbb{L}$, $\mathbb{C}$, $\mathbb{B}$ and $\mathbb{D}$ respectively, whose definitions are similar in the following sense. First, if $\mathbb{P} \in \{\mathbb{S}, \mathbb{M}, \mathbb{L}, \mathbb{C}\}$, then $\mathbb{P}$ can be represented as a set of subtrees of the reversed tree $\langle \mathcal{T}_{\mathbb{P}}, \supset\rangle$, where $\mathcal{T}_{\mathbb{P}} \in \{{}^{<\omega}2, {}^{<\omega}\omega\}$. Secondly, to any of these forcings, $\mathbb{P}$, we can adjoin a set of reals $\mathbb{R}_{\mathbb{P}} \in \{2^\omega, \omega^\omega, \omega^{\uparrow\omega}\}$ (where $\omega^{\uparrow\omega}$ denotes the set of all increasing functions from ω to ω) and a σ-ideal $\mathbb{I}_{\mathbb{P}}$ of subsets of $\mathbb{R}_{\mathbb{P}}$ such that the ordering $\mathbb{P}$, the complete Boolean algebra $\langle \mathrm{Borel}(\mathbb{R}_{\mathbb{P}})/\mathbb{I}_{\mathbb{P}}, \leq\rangle$ and the poset $\langle \mathrm{Borel}(\mathbb{R}_{\mathbb{P}}) \setminus \mathbb{I}_{\mathbb{P}}, \subset\rangle$ are forcing-equivalent orderings. As in [2] such forcings will be called *real forcings*. The situation is described by the following table (the notation will be explained in the sequel).

$\mathbb{P}$	$\mathcal{T}_{\mathbb{P}}$	a tree $T \subset \mathcal{T}_{\mathbb{P}}$ is an element of $\mathbb{P}$ iff	$\mathbb{R}_{\mathbb{P}}$	a subset $I \subset \mathbb{R}_{\mathbb{P}}$ belongs to the ideal $\mathbb{I}_{\mathbb{P}}$ iff
$\mathbb{S}$ (Sacks)	${}^{<\omega}2$	below each $\varphi \in T$ there is $\psi \in T$ having 2 predecessors in T	2^ω	I is countable
$\mathbb{M}$ (Miller)	${}^{<\omega}\omega$	below each $\varphi \in T$ there is $\psi \in T$ having $\aleph_0$ predecessors in T	ω^ω	I is $\leq^*$-bounded
$\mathbb{L}$ (Laver)	${}^{<\omega}\omega$	there is $\varphi_T \in T$ compatible with all $\psi \in T$, such that each $\varphi \in T \cap \varphi_T{\downarrow}$ has $\aleph_0$ predecessors in T	ω^ω	I is not strongly dominating
$\mathbb{C}$ (Cohen)	${}^{<\omega}2$	there is $\varphi_T \in T$ such that $T = \varphi_T{\uparrow} \cup \varphi_T{\downarrow}$	2^ω	I is meager
$\mathbb{B}$ (random)	–	–	2^ω	I is of measure zero
$\mathbb{D}$ (Hechler)	–	–	$\omega^{\uparrow\omega}$	I is meager in the space $\langle \omega^{\uparrow\omega}, \mathcal{O}_{\mathbb{D}}\rangle$

We remind the reader that if $f, g : \omega \to \omega$ then $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. The set $I \subset \omega^\omega$ is *$\leq^*$-bounded* iff there is $g \in \omega^\omega$ such that $f \leq^* g$ for all $f \in I$. The set $I \subset \omega^\omega$ is *$\leq^*$-dominating* iff for each $f \in \omega^\omega$ there is $g \in I$ such that $f \leq^* g$. The set $X \subset \omega^\omega$ is *strongly dominating* iff for each $F : {}^{<\omega}\omega \to \omega$ there exists $x \in X$ such that $F_x \leq^* x$, where the function $F_x : \omega \to \omega$ is defined by: $F_x(n) = F(x \restriction n)$, for all $n \in \omega$. If $\langle \mathbb{P}, \leq\rangle$ is a partial ordering and $p \in \mathbb{P}$, then $p{\downarrow} = \{q \in \mathbb{P} : q \leq p\}$ and $p{\uparrow} = \{q \in \mathbb{P} : p \leq q\}$. The topology $\mathcal{O}_D$ on the set $\omega^{\uparrow\omega}$ is described in [2]. Finally, 'The f.c.e.' abbreviates 'The following conditions are equivalent' and $f : X \xrightarrow{\text{fin-1}} Y$ denotes 'f is a finite-to-one function from X to Y'.

2. A characterization of stable splitting families

First, we introduce some notation and terminology and list some well known facts. Let $\mathcal{T}$ be the tree ${}^{<\omega}2$ or the tree ${}^{<\omega}\omega$ and let $\mathbb{R}_{\mathcal{T}}$ be the corresponding set of reals, i.e. $\mathbb{R}_{\mathcal{T}} = 2^\omega$ or $\mathbb{R}_{\mathcal{T}} = \omega^\omega$. For a real $x \in \mathbb{R}_{\mathcal{T}}$ let $\widetilde{x} = \{x \restriction n : n \in \omega\}$ be the corresponding branch in $\mathcal{T}$. Then for a subset $\mathcal{B} \subset \mathcal{T}$ let

$$G_\delta(\mathcal{B}) = \{x \in \mathbb{R}_{\mathcal{T}} : |\widetilde{x} \cap \mathcal{B}| = \aleph_0\}.$$

Clearly, the sets $[\varphi] = \{x \in \mathbb{R}_{\mathcal{T}} : \varphi \subset x\}$, $\varphi \in \mathcal{T}$, form a clopen base for the standard topology on $\mathbb{R}_{\mathcal{T}}$ and $G_\delta(\mathcal{B}) = \bigcap_{m\in\omega} \bigcup_{\varphi\in\mathcal{B}, |\varphi|\geq m} [\varphi]$ is a G_δ-set, so, $G_\delta(\mathcal{B})$ is a Borel subset of $\mathbb{R}_{\mathcal{T}}$. If $\mathbb{I} \subset P(\mathbb{R}_{\mathcal{T}})$ is an ideal, then a set $\mathcal{B} \subset \mathcal{T}$ will be called *positive* iff $G_\delta(\mathcal{B}) \notin \mathbb{I}$ and $\mathcal{T}^+$ will be the collection of all positive subsets of $\mathcal{T}$. According to [2], p. 278, for each forcing notion $\mathbb{P}$ considered here the set $\{G_\delta(\mathcal{B}) : \mathcal{B} \in \mathcal{T}_{\mathbb{P}}^+\}$ is dense in $\mathbb{P}$ and, moreover, there holds

$$\forall \mathcal{B} \in \mathcal{T}_{\mathbb{P}}^+ \ \ \forall p \leq G_\delta(\mathcal{B}) \ \ \exists \mathcal{B}' \in \mathcal{T}_{\mathbb{P}}^+ \ \ (\mathcal{B}' \subset \mathcal{B} \wedge G_\delta(\mathcal{B}') \leq p). \tag{$*$}$$

According to Lemmas 42.2 and 42.3 of [5] there holds

Fact 1. *Let $M \subset N$ be transitive models of ZFC. If in M the set B is a Borel subset of the Baire space ω^ω and c is a Borel code of B belonging to M and if by B^N we denote the N-evaluation of c, then:*

(a) *The evaluation B^N does not depend on the choice of the code c. Also, $B \subset B_1$ implies $B^N \subset B_1^N$.*

(b) *For Borel sets $B_1, B_2 \in M$ there holds $[B_1 \circ B_2]^N = B_1^N \circ B_2^N$, for $\circ \in \{\cup, \cap, \setminus\}$.*

(c) *If $\langle B_n : n \in \omega\rangle$ is a sequence of Borel sets belonging to M, then there holds $[\bigcup_{n\in\omega} B_n]^N = \bigcup_{n\in\omega} B_n^N$ and $[\bigcap_{n\in\omega} B_n]^N = \bigcap_{n\in\omega} B_n^N$.*

So, since $[G_\delta(\mathcal{B})]^N = \bigcap_{m\in\omega} \bigcup_{n\geq m} \bigcup_{\varphi\in\mathcal{B}\cap{}^n\omega} [\varphi]^N$ and $[\varphi]^N = [\varphi] \cap N$, we have:

Fact 2. *Let $M \subset N$ be transitive models of ZFC. If $\mathcal{B} \subset {}^{<\omega}\omega$ and $\mathcal{B} \in M$, then for each real $x \in \omega^\omega \cap N$ there holds: $x \in [G_\delta(\mathcal{B})]^N$ iff $|\widetilde{x} \cap \mathcal{B}| = \aleph_0$.*

Clearly, the previous two facts remain true if we replace the Baire space ω^ω by the Cantor cube 2^ω.

Fact 3 (*Zapletal, see [12] or [2], Lemma 2.1.1*). *If $\mathbb{P} = \mathrm{Borel}(\mathbb{R}_\mathbb{P}) \setminus \mathbb{I}_\mathbb{P}$ is a real forcing and G a $\mathbb{P}$-generic filter over V, then there is a real $x \in V[G]$ such that for each $B \in \mathrm{Borel}(\mathbb{R}_\mathbb{P})$ there holds:*

$$B \in G \Leftrightarrow x \in B^{V[G]},$$

where $B^{V[G]}$ is the $V[G]$-evaluation of a Borel code $c(\in V)$ of B.

If $\mathbb{P}$ is a forcing notion, then a $\mathbb{P}$-name of the form $\tau = \bigcup_{n\in\omega} \{\check{n}\} \times A_n$, where A_n are antichains in $\mathbb{P}$, is called a nice name for a subset of ω. Let $\mathrm{Nn}(\check{\omega})$ be the set of all such names and let $\mathrm{Nn}^\omega(\check{\omega}) = \{\tau \in \mathrm{Nn}(\check{\omega}) : 1_\mathbb{P} \Vdash |\tau| = \check{\omega}\}$. If $\tau \in \mathrm{Nn}(\check{\omega})$ and $r \in \mathbb{P}$, let $\tau_r = \{n \in \omega : \exists s \leq r \;\; s \Vdash \check{n} \in \tau\}$. The following standard fact will be used in the sequel.

Fact 4. *Let $\mathbb{P}$ be a forcing notion.*

(a) *If $\tau \in \mathrm{Nn}(\check{\omega})$, $p \in \mathbb{P}$ and $S \subset \omega$, then $p \Vdash |\tau \cap \check{S}| = \check{\omega}$ if and only if $\forall r \leq p \;\; |\tau_r \cap S| = \omega$.*

(b) *If φ is a formula of ZFC, then $1_\mathbb{P} \Vdash \forall X \in [\check{\omega}]^{\check{\omega}} \;\; \varphi(X, \ldots)$ if and only if $\forall \tau \in \mathrm{Nn}^\omega(\check{\omega}) \;\; 1_\mathbb{P} \Vdash \varphi(\tau, \ldots)$.*

The following property of real forcings, distinguished by Brendle and Yatabe in [2], is crucial for obtaining of our results.

Definition 1 (*Brendle and Yatabe [2]*). A real forcing $\mathbb{P} = \mathrm{Borel}(\mathbb{R}_\mathbb{P}) \setminus \mathbb{I}_\mathbb{P}$ has *weak fusion* if for each $\tau \in \mathrm{Nn}(\check{\omega})$ and each $p \in \mathbb{P}$, where $p \Vdash |\tau| = \check{\omega}$, there exist

(wf1) disjoint antichains $\mathcal{B}_n \subset \mathcal{T}_\mathbb{P}$, $n \in \omega$, such that $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ is a positive set and $G_\delta(\mathcal{B}) \leq p$;

(wf2) antichains $\mathcal{A}_n \subset \mathbb{P}$, $n \in \omega$;

(wf3) injections $h_n : \mathcal{B}_n \to \mathcal{A}_n$, $n \in \omega$, such that for each positive set $\mathcal{B}' \subset \mathcal{B}$ the set $M_{\mathcal{B}'} = \{n \in \omega : \exists \varphi \in \mathcal{B}_n \cap \mathcal{B}' \;\; ([\varphi] \cap G_\delta(\mathcal{B}') \cap h_n(\varphi) \notin \mathbb{I}_\mathbb{P})\}$ is infinite;

(wf4) an injection $g : \bigcup_{n\in\omega} \{n\} \times \mathcal{A}_n \to \omega$ such that for each $\langle n, s\rangle \in \mathrm{dom}(g)$ there holds $s \Vdash \check{g(n, s)} \in \tau \setminus \check{n}$.

If the condition 'g is one-to-one' is replaced by the condition 'g is finite-to-one', then the definition of *very weak fusion* is obtained.

Lemma 1. *Let $\mathbb{P} = \mathrm{Borel}(\mathbb{R}_\mathbb{P}) \setminus \mathbb{I}_\mathbb{P}$ be a real forcing satisfying weak fusion and ($*$), let $p \in \mathbb{P}$ and $\tau \in \mathrm{Nn}(\check{\omega})$, where $p \Vdash |\tau| = \check{\omega}$. If $\mathcal{B}_n, \mathcal{A}_n, h_n$ and g are the objects provided by weak fusion and $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$, then:*

(a) *The function $f : \mathcal{B} \to \omega$, where $f(\varphi) = g(n, h_n(\varphi))$ for $\varphi \in \mathcal{B}_n$, is an injection;*

(b) *If $S \subset \omega$ and $G_\delta(f^{-1}[S]) \notin \mathbb{I}_\mathbb{P}$, then $G_\delta(f^{-1}[S]) \leq p$ and $G_\delta(f^{-1}[S]) \Vdash |\tau \cap \check{S}| = \check{\omega}$.*

If instead of weak fusion the forcing $\mathbb{P}$ has very weak fusion, then the function f defined in (a) *is finite-to-one and* (b) *holds again.*

Proof. (a) Since the sets $\mathcal{B}_n$, $n \in \omega$, are disjoint, the function f is well-defined. Let $\varphi, \psi \in \mathcal{B}$, $\varphi \neq \psi$ and $\varphi \in \mathcal{B}_m$, $\psi \in \mathcal{B}_n$. If $m \neq n$, then $\langle m, h_m(\varphi)\rangle \neq \langle n, h_n(\varphi)\rangle$ and if $m = n$, then $\langle n, h_n(\varphi)\rangle \neq \langle n, h_n(\psi)\rangle$, because h_n is an injection. Now $f(\varphi) \neq f(\psi)$, since g is an injection too.

(b) Let $S \subset \omega$ and $G_\delta(f^{-1}[S]) \notin \mathbb{I}_\mathbb{P}$. The operator G_δ is monotone, so, since $f^{-1}[S] \subset \mathcal{B}$ and since (wf1) holds, we have $G_\delta(f^{-1}[S]) \leq p$. According to Fact 4 it remains to be proved that

$$\forall r \leq G_\delta(f^{-1}[S]) \;\; |\tau_r \cap S| = \omega. \tag{1}$$

Let $r \leq G_\delta(f^{-1}[S])$. By $(*)$, there exists $\mathcal{B}' \subset f^{-1}[S]$ such that $\mathbb{I}_\mathbb{P} \not\ni G_\delta(\mathcal{B}') \leq r$. By (wf3) the set $M_{\mathcal{B}'} = \{n \in \omega : \exists \varphi_n \in \mathcal{B}_n \cap \mathcal{B}' \ \ [\varphi_n] \cap G_\delta(\mathcal{B}') \cap h_n(\varphi_n) \notin \mathbb{I}_\mathbb{P}\}$ is infinite. We will prove that

$$\{f(\varphi_n) : n \in M_{\mathcal{B}'}\} \subset \tau_r \cap S. \tag{2}$$

Firstly, $\varphi_n \in \mathcal{B}' \subset f^{-1}[S]$, so $f(\varphi_n) \in S$, for all $n \in M_{\mathcal{B}'}$. Secondly, by (wf4) we have $h_n(\varphi_n) \Vdash f(\varphi_n)\check{} \in \tau$, so, for $t = [\varphi_n] \cap G_\delta(\mathcal{B}') \cap h_n(\varphi_n)$ there holds $t \Vdash f(\varphi_n)\check{} \in \tau$. Since $t \leq G_\delta(\mathcal{B}') \leq r$ we have $\exists t \leq r \ \ t \Vdash f(\varphi_n)\check{} \in \tau$, that is $f(\varphi_n) \in \tau_r$. Thus (2) is true and, since f is one-to-one, we have $|\tau_r \cap S| = \omega$. The proof of (1) is finished. □

Now, a characterization of stable splitting families for a wide class of real forcings follows:

Theorem 2. *Let $\mathbb{P} = \mathrm{Borel}(\mathbb{R}_\mathbb{P}) \setminus \mathbb{I}_\mathbb{P}$ be a real forcing satisfying weak fusion and $(*)$. If $\mathcal{S}$ is a splitting family on ω, then the f.c.e.:*
(a) *$\mathcal{S}$ is $\mathbb{P}$-stable;*
(b) $\forall \tau \in \mathrm{Nn}^\omega(\check{\omega}) \ \ \forall p \in \mathbb{P} \ \ \exists q \leq p \ \ \exists S \in \mathcal{S} \ \ q \Vdash |\tau \cap \check{S}| = \check{\omega} \wedge |\tau \setminus \check{S}| = \check{\omega}$;
(c) $\forall \mathcal{B} \in \mathcal{T}_\mathbb{P}^+ \ \ \forall f : \mathcal{B} \xrightarrow{1-1} \omega \ \ \exists S \in \mathcal{S} \ \ G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_\mathbb{P}$.
If instead of weak fusion the forcing $\mathbb{P}$ has very weak fusion, then conditions (a), (b) *and* (d) *are equivalent, where*
(d) $\forall \mathcal{B} \in \mathcal{T}_\mathbb{P}^+ \ \ \forall f : \mathcal{B} \xrightarrow{\textit{fin-1}} \omega \ \ \exists S \in \mathcal{S} \ \ G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_\mathbb{P}$.

Proof. The equivalence (a)⇔(b) is a consequence of Fact 4 and the elementary properties of the forcing relation.

(a)⇒(c). Suppose (a) and ¬(c). Then there are $\mathcal{B} \in \mathcal{T}_\mathbb{P}^+$ and $f : \mathcal{B} \xrightarrow{1-1} \omega$ such that

$$\forall S \in \mathcal{S} \ \ G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \in \mathbb{I}_\mathbb{P}. \tag{3}$$

Let G be a $\mathbb{P}$-generic filter over V such that $G_\delta(\mathcal{B}) \in G$. According to Fact 3 there exists a generic real $x \in V[G]$ and $x \in G_\delta(\mathcal{B})^{V(G)}$. Since $G \cap \mathbb{I}_\mathbb{P} = \emptyset$ using (3) and Fact 3 we obtain

$$\forall S \in \mathcal{S} \ \ x \notin [G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S])]^{V[G]}. \tag{4}$$

Now, since countable operations with Borel sets having Borel codes in V are absolute (see [5], Lemma 42.3) it follows that the $V[G]$-evaluation $G_\delta(\mathcal{B})^{V(G)}$ is equal to the set $G_\delta^{V(G)}(\mathcal{B}) = \{y \in \mathbb{R}_\mathbb{P}^{V[G]} : |\widetilde{y} \cap B| = \aleph_0\}$. By (4) we obtain

$$\forall S \in \mathcal{S} \ \ x \notin G_\delta^{V(G)}(f^{-1}[S]) \cap G_\delta^{V(G)}(f^{-1}[\omega \setminus S]). \tag{5}$$

Since $x \in G_\delta(\mathcal{B})^{V(G)}$, the set $\tilde{x} \cap \mathcal{B}$ is infinite, so, the set $X = f[\tilde{x} \cap \mathcal{B}]$ is infinite, because f is an injection. By (a) there is a set $S \in \mathcal{S}$ which splits X, that is $X \cap S = \{f(x \restriction n) : n \in N\}$ and $X \setminus S = \{f(x \restriction n) : n \in M\}$, where $N, M \in [\omega]^\omega$. So, for $n \in N$ we have $x \restriction n \in f^{-1}[S]$ thus $x \in G_\delta^{V[G]}(f^{-1}[S])$ and, similarly, $x \in G_\delta^{V[G]}(f^{-1}[\omega \setminus S])$. A contradiction to (5).

(c)⇒(b). Let condition (c) hold, $\tau \in \mathrm{Nn}^\omega(\check{\omega})$ and $p \in \mathbb{P}$. We will find $q \leq p$ and $S \in \mathcal{S}$ such that $q \Vdash |\tau \cap \check{S}| = \check{\omega} \wedge |\tau \setminus \check{S}| = \check{\omega}$. Let $\mathcal{B}_n, \mathcal{A}_n, h_n$ and g be the objects provided by weak fusion, let $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$ and let $f : \mathcal{B} \to \omega$ be defined by $f(\varphi) = g(n, h_n(\varphi))$, for $\varphi \in \mathcal{B}_n$. By Lemma 1, f is an injection. Since $G_\delta(\mathcal{B}) \notin \mathbb{I}_\mathbb{P}$, using (c) we obtain $S \in \mathcal{S}$ such that $q \stackrel{df}{=} G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_\mathbb{P}$. By (b) of Lemma 1, $q \leq p$, $G_\delta(f^{-1}[S]) \Vdash |\tau \cap \check{S}| = \check{\omega}$ and, consequently, $q \Vdash |\tau \cap \check{S}| = \check{\omega}$. Similarly, by Lemma 1, $G_\delta(f^{-1}[\omega \setminus S]) \Vdash |\tau \setminus \check{S}| = \check{\omega}$, so $q \Vdash |\tau \setminus \check{S}| = \check{\omega}$ too. □

By [2], p. 278, the forcings $\mathbb{S}, \mathbb{M}, \mathbb{L}, \mathbb{C}, \mathbb{D}$ and $\mathbb{B}$ satisfy condition $(*)$ and $\mathbb{S}, \mathbb{M}, \mathbb{L}, \mathbb{C}$ and $\mathbb{D}$ have weak fusion (see Lemmas 2.2.3, 2.2.4 and 2.2.5 of [2]), so using the previous theorem we obtain:

Corollary 1. *Let $\mathbb{P} \in \{\mathbb{S}, \mathbb{M}, \mathbb{L}, \mathbb{C}, \mathbb{D}\}$ and let $\mathcal{S}$ be a splitting family on ω. Then $\mathcal{S}$ is $\mathbb{P}$-stable iff*

$$\forall \mathcal{B} \in \mathcal{T}_\mathbb{P}^+ \ \ \forall f : \mathcal{B} \xrightarrow{1-1} \omega \ \ \exists S \in \mathcal{S} \ \ G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_\mathbb{P}.$$

By Lemma 2.3.1 of [2] the random forcing has very weak fusion, so there holds

Corollary 2. *A splitting family $\mathcal{S}$ on ω is $\mathbb{B}$-stable iff*

$$\forall \mathcal{B} \in \mathcal{T}_\mathbb{B}^+ \ \ \forall f : \mathcal{B} \xrightarrow{\textit{fin-1}} \omega \ \ \exists S \in \mathcal{S} \ \ G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_\mathbb{B}.$$

Remark 1. In contrast to the characterization of stable tall ideals and mad families given in Theorem 2.2.2 of [2], in (c) of Theorem 2 we can not replace the part '$\forall f : \mathcal{B} \stackrel{1-1}{\longrightarrow} \omega$' by '$\forall f : \mathcal{B} \longrightarrow \omega$' since the formula

$$\forall f : \mathcal{B} \longrightarrow \omega \;\; \exists S \in \mathcal{S} \;\; G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_\mathbb{P}$$

is always false (take a constant function f).

3. The G_δ-homogeneity

In order to simplify characterizations of stable splitting families for some forcing notions, by the following definition we introduce a notion of homogeneity (of an ideal on the set of reals) which is slightly different from the notion of strong homogeneity used by Brendle and Yatabe in [2]

Definition 2. Let $\mathbb{P} = \mathrm{Borel}(\mathbb{R}_\mathbb{P}) \setminus \mathbb{I}_\mathbb{P}$ be a real forcing. The ideal $\mathbb{I}_\mathbb{P}$ is G_δ-*homogeneous* iff for each positive subset $\mathcal{B}$ of $\mathcal{T}_\mathbb{P}$ there is an embedding $h : \mathcal{T}_\mathbb{P} \to \mathcal{B}$ which preserves positive sets, i.e. h is an injection satisfying:
(h1) $\forall \varphi, \psi \in \mathcal{T}_\mathbb{P} \;\; (\varphi < \psi \Leftrightarrow h(\varphi) < h(\psi))$;
(h2) $\forall \mathcal{C} \in \mathcal{T}_\mathbb{P}^+ \;\; h[\mathcal{C}] \in \mathcal{T}_\mathbb{P}^+$.

Lemma 2. *Under the assumptions of the previous definition, for each* $\mathcal{C}, \mathcal{D} \subset \mathcal{T}_\mathbb{P}$

$$G_\delta(\mathcal{C}) \subset G_\delta(\mathcal{D}) \Rightarrow G_\delta(h[\mathcal{C}]) \subset G_\delta(h[\mathcal{D}]).$$

Proof. Let $\mathcal{C}, \mathcal{D} \subset \mathcal{T}_\mathbb{P}$ and $G_\delta(\mathcal{C}) \subset G_\delta(\mathcal{D})$. Let $y \in G_\delta(h[\mathcal{C}])$, that is $\{y \restriction k : k \in M\} \subset h[\mathcal{C}]$, where $M \in [\omega]^\omega$. Then there are $\varphi_k \in \mathcal{C}, \;\; k \in M$, such that $y \restriction k = h(\varphi_k)$. By (h1) $\{\varphi_k : k \in M\}$ is an infinite chain in $\mathcal{C}$, so $x = \bigcup_{k \in M} \varphi_k \in \mathbb{R}_\mathbb{P}$ and, clearly, $x \in G_\delta(\mathcal{C})$. By the assumption $|\tilde{x} \cap \mathcal{D}| = \aleph_0$, so, since h is an injection, the set $h[\tilde{x}] \cap h[\mathcal{D}]$ is infinite and $y \in G_\delta(h[\mathcal{D}])$ will follow from $h[\tilde{x}] \subset \tilde{y}$. If $l \in \omega$, then $l \in \mathrm{dom}(\varphi_k)$ for some $k \in M$, so $x \restriction l = \varphi_k \restriction l$. By (h1), $\varphi_k \restriction l > \varphi_k$ implies $h(x \restriction l) = h(\varphi_k \restriction l) > h(\varphi_k) = y \restriction k$ which means that for some $r < k$, $h(x \restriction l) = y \restriction r \in \tilde{y}$. Thus $h[\tilde{x}] \subset \tilde{y}$. $\square$

Theorem 3. *Let* $\mathbb{P} = \mathrm{Borel}(\mathbb{R}_\mathbb{P}) \setminus \mathbb{I}_\mathbb{P}$ *be a real forcing, satisfying weak fusion and* ($*$)*, where the ideal* $\mathbb{I}_\mathbb{P}$ *is* G_δ-*homogeneous. If* $\mathcal{S}$ *is a splitting family on* ω*, then the f.c.e.:*
(a) $\mathcal{S}$ *is* $\mathbb{P}$-*stable;*
(c′) $\forall f : \mathcal{T}_\mathbb{P} \stackrel{1-1}{\longrightarrow} \omega \;\; \exists S \in \mathcal{S} \;\; G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_\mathbb{P}$.

Proof. We prove (c′) $\Leftrightarrow$ (c) of Theorem 2. (c) $\Rightarrow$ (c′) is true since $\mathcal{T}_\mathbb{P} \in \mathcal{T}_\mathbb{P}^+$.

(c′) $\Rightarrow$ (c). Suppose (c′). Let $\mathcal{B} \subset \mathcal{T}_\mathbb{P}$ be a positive set and $f : \mathcal{B} \stackrel{1-1}{\to} \omega$. Let $h : \mathcal{T}_\mathbb{P} \to \mathcal{B}$ be an embedding preserving positive sets. Then $f \circ h : \mathcal{T}_\mathbb{P} \to \omega$ is an injection, so, by (c′), there is $S \in \mathcal{S}$ such that

$$p = G_\delta(h^{-1}[f^{-1}[S]]) \cap G_\delta(h^{-1}[f^{-1}[\omega \setminus S]]) \notin \mathbb{I}_\mathbb{P}.$$

Consequently, the sets $h^{-1}[f^{-1}[S]])$ and $h^{-1}[f^{-1}[\omega \setminus S]]$ are positive. Since $p \subset G_\delta(h^{-1}[f^{-1}[S]])$, using ($*$) we obtain a positive set $\mathcal{D} \subset h^{-1}[f^{-1}[S]]$ such that $G_\delta(\mathcal{D}) \subset p$. Now $G_\delta(\mathcal{D}) \subset G_\delta(h^{-1}[f^{-1}[\omega \setminus S]])$ and, using ($*$) again, we obtain a positive set $\mathcal{C} \subset h^{-1}[f^{-1}[\omega \setminus S]]$ such that $G_\delta(\mathcal{C}) \subset G_\delta(\mathcal{D})$ which, by Lemma 2 implies

$$G_\delta(h[\mathcal{C}]) \subset G_\delta(h[\mathcal{D}]). \tag{6}$$

Clearly $h[\mathcal{C}] \subset f^{-1}[\omega \setminus S]$, thus $G_\delta(h[\mathcal{C}]) \subset G_\delta(f^{-1}[\omega \setminus S])$. Also, $h[\mathcal{D}] \subset f^{-1}[S]$, thus $G_\delta(h[\mathcal{D}]) \subset G_\delta(f^{-1}[S])$ so, by (6),

$$G_\delta(h[\mathcal{C}]) \subset G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]). \tag{7}$$

Since the set $\mathcal{C}$ is positive, $h[\mathcal{C}]$ is positive too and, by (7), $G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_\mathbb{P}$. $\square$

4. Additional characterizations of $\mathbb{S}$-stability

Lemma 3. *If $\mathcal{B} \subset {}^{<\omega}2$ and $|G_\delta(\mathcal{B})| > \aleph_0$, then:*

(a) *The set $\mathcal{B}' = \{\varphi \in \mathcal{B} : |[\varphi] \cap G_\delta(\mathcal{B})| > \aleph_0\}$ is nonempty.*

(b) *Below each $\varphi \in \mathcal{B}'$ there are two incompatible elements $\psi^0, \psi^1 \in \mathcal{B}'$.*

Proof. (a) follows from $G_\delta(\mathcal{B}) \subset \bigcup_{\varphi \in \mathcal{B}} [\varphi]$ and $|\mathcal{B}| = \aleph_0$.

(b) Let $\varphi \in \mathcal{B}'$. Then $A = [\varphi] \cap G_\delta(\mathcal{B})$ is an uncountable subset of the Cantor cube, 2^ω, and (see [3], 1.7.11) $|A \setminus A^0| \leq \aleph_0$, where $A^0 = \{x \in 2^\omega : |U \cap A| > \aleph_0$ for each neighbourhood U of $x\}$. Thus $|A \cap A^0| > \aleph_0$, so we can choose different points $x, y \in A \cap A^0$. Let $n_0 = \min\{n \in \omega : x(n) \neq y(n)\}$, then clearly $\varphi \subset x \upharpoonright n_0 = y \upharpoonright n_0$. Since $x, y \in G_\delta(\mathcal{B})$ there are $\psi^0, \psi^1 \in \mathcal{B}$ such that $\psi^0 \subset x$, $\psi^1 \subset y$ and $|\psi^0|, |\psi^1| > n_0$. Then $\psi^0 \perp \psi^1$ and clearly $\psi^0, \psi^1 < \varphi$. Since $x \in A^0$ and $[\psi^0]$ is a neighborhood of x, the set $[\psi^0] \cap A = [\psi^0] \cap G_\delta(\mathcal{B})$ is uncountable, so $\psi^0 \in \mathcal{B}'$. Analogously, $\psi^1 \in \mathcal{B}'$. $\square$

Theorem 4. *Let $\mathcal{S}$ be a splitting family on ω. Then the f.c.e.:*

($\mathbb{S}$1) *$\mathcal{S}$ is $\mathbb{S}$-stable;*

($\mathbb{S}$2) $\forall f : {}^{<\omega}2 \xrightarrow{1-1} \omega \ \exists S \in \mathcal{S} \ \ |G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S])| > \aleph_0$;

($\mathbb{S}$3) *$\mathcal{S}$ is stable for some forcing adding a new real.*

Proof. ($\mathbb{S}$1)$\Leftrightarrow$($\mathbb{S}$2). Since the Sacks forcing has weak fusion ([2], Lemma 2.2.3) and satisfies condition ($*$), according to Theorem 3, it remains to be shown that the ideal $\mathbb{I}_\mathbb{S} = [2^\omega]^{\leq\omega}$ is G_δ-homogeneous. Let $\mathcal{B} \subset {}^{<\omega}2$, where $|G_\delta(\mathcal{B})| > \aleph_0$ and

$$\mathcal{B}' = \{\psi \in \mathcal{B} : |[\psi] \cap G_\delta(\mathcal{B})| > \aleph_0\}.$$

Using Lemma 3, below each $\psi \in \mathcal{B}'$ we choose incompatible $\eta^0_\psi, \eta^1_\psi \in \mathcal{B}'$ and by recursion we define $h : {}^{<\omega}2 \to \mathcal{B}'$ by

- $h(\emptyset) = \varphi_\emptyset$, an arbitrary element of $\mathcal{B}'$;
- if $h(\varphi)$ is defined, then $h(\varphi^\frown 0) = \eta^0_{h(\varphi)}$ and $h(\varphi^\frown 1) = \eta^1_{h(\varphi)}$.

Now, using induction, we easily prove that for each $n \in \omega$ the restriction of h to $2^{\leq n}$ is one-to-one and that $\varphi < \psi \Leftrightarrow h(\varphi) < h(\psi)$, for all $\varphi, \psi \in 2^{\leq n}$. Thus h is an injection and condition (h1) of Definition 2 is satisfied.

In order to prove (h2) suppose $\mathcal{C} \subset {}^{<\omega}2$ and $|G_\delta(\mathcal{C})| > \aleph_0$. We prove $|G_\delta(h[\mathcal{C}])| > \aleph_0$. If $x \in G_\delta(\mathcal{C})$, that is $\{x \upharpoonright k : k \in M\} \subset \mathcal{C}$ for some $M \in [\omega]^\omega$, then, by (h1), $\{h(x \upharpoonright k) : k \in M\}$ is an infinite chain in $h[\mathcal{C}]$ and $y_x = \bigcup_{k \in M} h(x \upharpoonright k) \in G_\delta(h[\mathcal{C}])$. If $x_1 \in G_\delta(\mathcal{C})$ and $x_1 \neq x$, then $x \upharpoonright l \perp x_1 \upharpoonright l$ for some $l \in \omega$ and, by (h1), $h(x \upharpoonright l) \perp h(x_1 \upharpoonright l)$, which implies $y_x \neq y_{x_1}$. Thus, $x \mapsto y_x$ is an one-to-one mapping from $G_\delta(\mathcal{C})$ into $G_\delta(h[\mathcal{C}])$, so $|G_\delta(h[\mathcal{C}])| > \aleph_0$. Thus $\mathbb{I}_\mathbb{S}$ is a G_δ-homogeneous ideal.

The implication ($\mathbb{S}$1)$\Rightarrow$($\mathbb{S}$3) is trivial and we prove ($\mathbb{S}$3)$\Rightarrow$($\mathbb{S}$2). Let $\mathbb{P}$ be a forcing adding a new real $x \in 2^\omega \cap V[G]$ and $V[G] \models$ "$\mathcal{S}$ is a splitting family". Suppose there is an injection $f : {}^{<\omega}2 \to \omega$, where

$$\forall S \in \mathcal{S} \ \ |G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S])| \leq \aleph_0.$$

Since a new real can not belong to a countable Borel set coded in V, we have

$$\forall S \in \mathcal{S} \ \ x \notin (G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]))^{V[G]}. \tag{8}$$

Since f is an injection, $f[\tilde{x}] \in [\omega]^\omega$. By the assumption there is a set $S \in \mathcal{S}$ such that $|f[\tilde{x}] \cap S| = \aleph_0$ and $|f[\tilde{x}] \setminus S| = \aleph_0$. Clearly, the set $\tilde{x} \cap f^{-1}[S]$ is infinite, so $x \in G_\delta^{V[G]}(f^{-1}[S])$. Also $|f^{-1}[f[\tilde{x}] \setminus S]| = |\tilde{x} \cap f^{-1}[\omega \setminus S]| = \aleph_0$, so $x \in G_\delta^{V[G]}(f^{-1}[\omega \setminus S])$. By the argument concerning V-codes of Borel sets given in the proof of Theorem 2 we obtain a contradiction to (8). Thus ($\mathbb{S}$2) holds. $\square$

5. Additional characterizations of $\mathbb{M}$-stability

An element ψ of a Miller tree $T \subset {}^{<\omega}\omega$ is called an *$\aleph_0$-splitting node of T* if the set $\{n \in \omega : \psi^\frown n \in T\}$ is infinite. Let $\mathrm{Split}(T)$ denote the set of all such nodes. According to [7], a subset P of the Baire space ω^ω will be called *superperfect* iff it is closed and the tree $T_P = \bigcup_{x \in P} \widetilde{x}$ is a Miller tree. By [7] (see also [12]) we have

Fact 5 (*Kechris [7]*). *Each analitic, $\leq^*$-unbounded subset of ω^ω has a superperfect subset.*

Lemma 4. *If $\mathcal{B} \in \mathcal{T}^+_{\mathbb{M}}$, then*
(a) *There is a superperfect set $P \subset G_\delta(\mathcal{B})$.*
(b) *Below each $\varphi \in \mathcal{B} \cap T_P$ there is $\psi \in \mathrm{Split}(T_P)$, and if $\{n_k : k \in \omega\}$ is an enumeration of the set $\{n \in \omega : \psi^\frown n \in T_P\}$, then for each $k \in \omega$ there exists $\varphi_k \in \mathcal{B} \cap T_P$ such that $\varphi_k \leq \psi^\frown n_k$.*

Proof. (a) By the assumption, $G_\delta(\mathcal{B})$ is an $\leq^*$-unbounded set and we apply Fact 5.
(b) Since T_P is a Miller tree, there is an $\psi \in \mathrm{Split}(T_P)$ such that $\psi \leq \varphi$. Since $\psi^\frown n_k \in T_P$, there is $x \in P$ such that $\psi^\frown n_k \subset x$. Now $x \in G_\delta(\mathcal{B})$ thus there is $\varphi_k \in \mathcal{B}$ satisfying $\psi^\frown n_k \subset \varphi_k \subset x$ which implies $\varphi_k \in T_P$ and $\varphi_k \leq \psi^\frown n_k$. $\square$

Lemma 5. *The ideal $\mathbb{I}_{\mathbb{M}}$ is G_δ- homogeneous.*

Proof. Let $\mathcal{B} \in \mathcal{T}^+_{\mathbb{M}}$. By Lemma 4(a), there is a superperfect set $P \subset G_\delta(\mathcal{B})$. Using recursion we define a function $h : {}^{<\omega}\omega \to \mathcal{B} \cap T_P$ as follows.
$h(\emptyset)$ is an arbitrary element of $\mathcal{B} \cap T_P$.
Let $n \in \omega$, $\eta \in {}^n\omega$ and let $h(\eta) \in \mathcal{B} \cap T_P$ be defined. According to Lemma 4(b) there is $\psi_\eta \in \mathrm{Split}(T_P)$ such that $\psi_\eta \leq h(\eta)$ and, if $\{n^\eta_k : k \in \omega\}$ is an increasing enumeration of the set $M_\eta = \{n \in \omega : \psi_\eta^\frown n \in T_P\}$, there are $\varphi^\eta_k \in \mathcal{B} \cap T_P$, $k \in \omega$, satisfying $\varphi^\eta_k \leq \psi_\eta^\frown n^\eta_k$. Now we define $h(\eta^\frown k) = \varphi^\eta_k$, for $k \in \omega$. So,

$$h(\eta^\frown k) \leq \psi_\eta^\frown n^\eta_k < \psi_\eta \leq h(\eta).$$

It is easy to show that h is an injection satisfying condition (h1) of Definition 2.
For a proof of (h2) suppose $\mathcal{C} \subset {}^{<\omega}\omega$ and $G_\delta(\mathcal{C})$ is an $\leq^*$-unbounded subset of ω^ω. We prove that the set $G_\delta(h[\mathcal{C}])$ is $\leq^*$-unbounded too, showing that for an arbitrary function $f : \omega \to \omega$ there is $y \in G_\delta(h[\mathcal{C}])$ which is not $\leq^*$-bounded by f. By Lemma 4(a) there is a superperfect set $Q \subset G_\delta(\mathcal{C})$. Using recursion we construct a sequence $\eta_0 \subset \eta_1 \subset \dots$ of elements of $\mathcal{C} \cap T_Q$ and an increasing sequence $r_1 < r_2 < \dots$ in ω such that for each $n \geq 1$

$$r_n \in \mathrm{dom}(h(\eta_n)) \text{ and } h(\eta_n)(r_n) > f(r_n). \tag{9}$$

Let $\eta_0 \in \mathcal{C} \cap T_Q$ be arbitrary.
Let $n \in \omega$ and let $\eta_0, \dots, \eta_n$ and $r_1, \dots, r_n$ be defined. Since $\eta_n \in \mathcal{C} \cap T_Q$, according to Lemma 4(b) there is $\nu_n \in \mathrm{Split}(T_Q)$ such that $\nu_n \leq \eta_n$ and, if $\{m^n_l : l \in \omega\}$ is an increasing enumeration of the set $N_{\nu_n} = \{m \in \omega : \nu_n^\frown m \in T_Q\}$, there are $\eta^n_l \in \mathcal{C} \cap T_Q$, $l \in \omega$, satisfying $\eta^n_l \leq \nu_n^\frown m^n_l$. Then $h(\eta^n_l) \in h[\mathcal{C}] \cap T_P$ and, clearly,

$$h(\eta^n_l) \leq h(\nu_n^\frown m^n_l) < h(\nu_n) \leq h(\eta_n).$$

According to the definition of h, there exists $\psi_{\nu_n} \in \mathrm{Split}(T_P)$ such that for each $l \in \omega$

$$h(\nu_n^\frown m^n_l) < \psi_{\nu_n} \leq h(\nu_n)$$

and for each $l \in \omega$ there is $k^n_l \in \omega$ such that

$$h(\nu_n^\frown m^n_l) \leq \psi_{\nu_n}^\frown k^n_l$$

and $l \neq l_1$ implies $k^n_l \neq k^n_{l_1}$. Let us define $r_{n+1} = |\psi_{\nu_n}| + 1$. Now we choose $l_n \in \omega$ such that $k^n_{l_n} > f(r_{n+1})$, which implies $h(\eta^n_{l_n})(r_{n+1}) = k^n_{l_n} > f(r_{n+1})$. Finally, we define $\eta_{n+1} = \eta^n_{l_n}$. Clearly (9) holds for $n + 1$. By (h1), $\langle h(\eta_n)\rangle$ is an $\subset$- increasing sequence in ${}^{<\omega}\omega$, thus $y = \bigcup_{n\in\omega} h(\eta_n)$ is a function from ω to ω. By the construction $\eta_n \in \mathcal{C}$, so $h(\eta_n) \in \widetilde{y} \cap h[\mathcal{C}]$, $n \in \omega$, thus $y \in G_\delta(h[\mathcal{C}])$. By (9), for each $n \geq 1$ there holds $y(r_n) > f(r_n)$ so y is not $\leq^*$-bounded by f. $\square$

Fact 6. *Let $M \subset N$ be transitive models of ZFC. If a Borel set $B \in M$ is $\leq^*$-bounded by a function $f \in \omega^\omega \cap M$, then B^N is $\leq^*$-bounded by the same function. Consequently, if N contains an unbounded real $x \in \omega^\omega$, then $x \notin B^N$.*

Proof. By the assumption, $B \subset \bigcup_{m\in\omega}\bigcap_{n\geq m}\bigcup_{\varphi\in {}^n\omega}\bigcup_{k\leq f(n)}[\varphi^\frown k] = B_1$ and, applying Fact 1, we obtain $B^N \subset \bigcup_{m\in\omega}\bigcap_{n\geq m}\bigcup_{\varphi\in {}^n\omega}\bigcup_{k\leq f(n)}[\varphi^\frown k]^N$. Now, since $[\varphi^\frown k]^N = [\varphi^\frown k]\cap N$, each real $x \in B^N$ is $\leq^* f$. □

Theorem 5. *If $\mathcal{S}$ is a splitting family on ω, then the f.c.e.:*

($\mathbb{M}$1) *$\mathcal{S}$ is $\mathbb{M}$-stable;*

($\mathbb{M}$2) $\forall f : {}^{<\omega}\omega \xrightarrow{1-1} \omega\ \exists S \in \mathcal{S}\ \ G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega\setminus S]) \notin \mathbb{I}_\mathbb{M}$*;*

($\mathbb{M}$3) *$\mathcal{S}$ is stable for some forcing adding an unbounded real.*

Proof. ($\mathbb{M}$1)⇔($\mathbb{M}$2). Since the Miller forcing has weak fusion ([2], Lemma 2.2.3) and satisfies condition (∗), according to Theorem 3, it remains to be shown that the ideal $\mathbb{I}_\mathbb{M}$ is G_δ-homogeneous. But this is Lemma 5.

($\mathbb{M}$1)⇒($\mathbb{M}$3). By [11], $\mathbb{M}$ adds unbounded reals.

($\mathbb{M}$3)⇒($\mathbb{M}$2). Let $\mathbb{P}$ be a forcing and $V_\mathbb{P}[G]$ an extension containing an unbounded real $x \in \omega^\omega$ while $V_\mathbb{P}[G] \models$ "$\mathcal{S}$ is a splitting family".

Let $f : {}^{<\omega}\omega \xrightarrow{1-1} \omega$ belong to V. Suppose that, in V, for each $S \in \mathcal{S}$ there holds $G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega\setminus S]) \in \mathbb{I}_\mathbb{M}$, that is $G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega\setminus S])$ is an $\leq^*$-bounded set. Then, by Fact 6, $x \notin [G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega\setminus S])]^{V_\mathbb{P}[G]}$ for all $S \in \mathcal{S}$ and, by Fact 1(b) we have

$$\forall S \in \mathcal{S}\ \ x \notin [G_\delta(f^{-1}[S])]^{V_\mathbb{P}[G]} \cap [G_\delta(f^{-1}[\omega\setminus S])]^{V_\mathbb{P}[G]}. \tag{10}$$

Since f is an injection, $f[\tilde{x}] \in [\omega]^\omega \cap V_\mathbb{P}[G]$ so, since $V_\mathbb{P}[G] \models$ '$\mathcal{S}$ is a splitting family', there is $S \in \mathcal{S}$ such that the sets $f[\tilde{x}] \cap S$ and $f[\tilde{x}] \cap \omega \setminus S$ are infinite. Then, clearly, the sets $\tilde{x} \cap f^{-1}[S]$ and $\tilde{x} \cap f^{-1}[\omega\setminus S]$ are infinite too thus, by Fact 2, $x \in [G_\delta(f^{-1}[S])]^{V_\mathbb{P}[G]} \cap [G_\delta(f^{-1}[\omega\setminus S])]^{V_\mathbb{P}[G]}$. A contradiction to (10). □

6. Additional characterizations of $\mathbb{L}$-stability and $\mathbb{C}$-stability

We remind the reader that the Laver forcing, $\langle \mathbb{L}, \subset\rangle$, is equivalent to the forcing $\mathrm{Borel}(\omega^\omega) \setminus \mathbb{I}_\mathbb{L}$, where $\mathbb{I}_\mathbb{L}$ is the σ-ideal of subsets of ω^ω which are not strongly dominating. This ideal is Borel-generated, moreover the following well-known fact holds.

Fact 7. *For a set of reals $X \subset \omega^\omega$ the f.c.e.:*

(i) $X \in \mathbb{I}_\mathbb{L}$*;*

(ii) *There is a function $F : {}^{<\omega}\omega \to \omega$ such that for each $x \in X$ the set $\{n \in \omega : F(x \restriction n) > x(n)\}$ is infinite;*

(iii) *There is a function $F : {}^{<\omega}\omega \to \omega$ such that $X \subset B_F$, where the Borel set B_F is given by $B_F = \bigcap_{m\in\omega}\bigcup_{n\geq m}\bigcap_{\varphi\in {}^n\omega}\bigcup_{k<F(\varphi)}[\varphi]^c \cup [\varphi^\frown k]$.*

If $T \subset {}^{<\omega}\omega$ is a tree, let $\mathrm{Br}(T) = \{x \in \omega^\omega : \tilde{x} \subset T\}$.

Fact 8. *If $T \subset {}^{<\omega}\omega$ is a Laver tree, then* $\mathrm{Br}(T) \in \mathrm{Borel}(\omega^\omega) \setminus \mathbb{I}_\mathbb{L}$.

Proof. Since $\mathrm{Br}(T) = \bigcap_{n\in\omega}\bigcup_{\varphi\in T\cap {}^n\omega}[\varphi]$, we have $\mathrm{Br}(T) \in \mathrm{Borel}(\omega^\omega)$. If $F : {}^{<\omega}\omega \to \omega$, we construct a real $x \in \mathrm{Br}(T)$ such that $F_x \leq^* x$. For $n < m = \mathrm{dom}\,\varphi_T$, let $x(n) = \varphi_T(n)$. Let $n \geq m$ and let $x \restriction n \in T$ be defined. Then we choose $k \in \omega$ such that $x \restriction n^\frown k \in T$ and $k \geq F(x \restriction n)$ and we define $x(n) = k$. So $F(x \restriction n) \leq x(n)$, for all $n \geq m$. □

Moreover, by [13] there holds:

Fact 9. *If $X \subset \omega^\omega$ is an analytic set, then $X \notin \mathbb{I}_\mathbb{L}$ iff there is a Laver tree T such that* $\mathrm{Br}(T) \subset X$.

The ideal $\mathbb{I}_\mathbb{L}$ is absolute in the following sense:

Fact 10. *Let $M \subset N$ be transitive models of ZFC. If, in M, a Borel set $B \subset \omega^\omega$ is not strongly dominating and $F : {}^{<\omega}\omega \to \omega$ is a witness for this, then, in N, the set B^N is not strongly dominating and F witnesses it too.*

Proof. By Facts 7 and 1, $B^N \subset \bigcap_{m\in\omega}\bigcup_{n\geq m}\bigcap_{\varphi\in {}^n\omega}\bigcup_{k<F(\varphi)}(\omega^\omega)^N \setminus [\varphi]^N \cup [\varphi^\frown k]^N$ so for each $x \in B^N$ we have $F_x \not\leq^* x$. □

We remind the reader that, if $M \subset N$ are models of ZFC, then a real $y \in \omega^\omega \cap N$ is *dominating over* M iff $f \leq^* y$, for each $f \in \omega^\omega \cap M$. A real y is *strongly dominating over* M iff $F_y \leq^* y$, for each function $F : {}^{<\omega}\omega \to \omega$ belonging to M (where $F_y(n) = F(y \restriction n)$).

Fact 11. *If $M \subset N$ are models of ZFC and N contains a dominating real, then N contains a strongly dominating real too.*

Proof. Let N contain a dominating real. Then, in N, there is a function $Y : {}^{<\omega}\omega \to \omega$ which $\leq^*$-dominates all functions $F : {}^{<\omega}\omega \to \omega$ belonging to M. If the function $y \in \omega^\omega \cap N$ is defined recursively by $y(n) = Y(y \restriction n)$, then for each $F : {}^{<\omega}\omega \to \omega$ belonging to M we have $F(y \restriction n) \leq y(n)$, for all but finitely many $n \in \omega$. □

Fact 12. *Let $M \subset N$ be models of ZFC. If in M B is a Borel subset of ω^ω and if N contains a strongly dominating real y, then:*
(a) *If $B \in \mathbb{I}_{\mathbb{L}}$, then $y \notin B^N$.*
(b) *If $B \notin \mathbb{I}_{\mathbb{L}}$, then there is a strongly dominating real $z \in B^N$.*

Proof. (a) is a consequence of Fact 10.

(b) Let, in M, $B \in \mathrm{Borel}(\omega^\omega) \setminus \mathbb{I}_{\mathbb{L}}$. By Fact 9, there is a Laver tree $T \subset {}^{<\omega}\omega$ such that $\mathrm{Br}(T) \subset B$. Let φ_T be the stem of T and $\mathrm{dom}(\varphi_T) = m$. Then the natural isomorphism $h : {}^{<\omega}\omega \to T \cap \varphi_T\!\downarrow$ is given by: $h(\emptyset) = \varphi_T$ and $h(\eta^\frown k) = h(\eta)^\frown n^\eta_k$, where $\{n^\eta_k : k \in \omega\}$ is an increasing enumeration of the set $M_\eta = \{n \in \omega : h(\eta)^\frown n \in T\}$.

Since $\{h(y \restriction n) : n \in \omega\}$ is a chain in T, for $z = \bigcup_{n\in\omega} h(y \restriction n)$ we have $z \in \omega^\omega \cap N$ and $\widetilde{z} \subset T$. According to Facts 8 and 1 there holds $\mathrm{Br}(T)^N = \{x \in \omega^\omega \cap N : \widetilde{x} \subset T\}$, thus $z \in \mathrm{Br}(T)^N$. By Fact 1, $\mathrm{Br}(T) \subset B$ implies $\mathrm{Br}(T)^N \subset B^N$ so, $z \in B^N$.

It remains to be proved that the function z is strongly dominating over M. Let $\mathrm{dom}(\varphi_T) = m$. Using induction we easily prove

$$\forall k \in \omega \ \left(\mathrm{dom}(h(y \restriction k)) = m + k \quad \text{and} \quad z(m + k) = n^{y\restriction k}_{y(k)}\right). \tag{11}$$

Since the enumerations of the sets M_η are increasing, we have $n^\eta_k \geq k$, for all $k \in \omega$, so, by (11),

$$\forall k \in \omega \ \left(z \restriction (m + k) = h(y \restriction k) \quad \text{and} \quad z(m + k) \geq y(k)\right). \tag{12}$$

Let $G : {}^{<\omega}\omega \to \omega$ and $G \in M$. We have to prove that $G(z \restriction l) \leq z(l)$ for almost all $l \in \omega$. Clearly, the function $F : {}^{<\omega}\omega \to \omega$ defined by $F(\varphi) = G(h(\varphi))$, $\varphi \in {}^{<\omega}\omega$, belongs to M and, since the real y is strongly dominating over M, there is $k_0 \in \omega$ such that for each $k \geq k_0$ there holds $F(y \restriction k) \leq y(k)$, that is $G(h(y \restriction k)) \leq y(k)$, which, according to (12), implies $G(z \restriction (m+k)) \leq y(k) \leq z(m+k)$. Thus, for each $l \geq m+k_0$ we have $G(z \restriction l) \leq z(l)$. □

Theorem 6. *If $\mathcal{S}$ is a splitting family on ω, then the f.c.e.:*
($\mathbb{L}$1) *$\mathcal{S}$ is $\mathbb{L}$-stable;*
($\mathbb{L}$2) $\forall \mathcal{B} \in \mathcal{T}^+_{\mathbb{L}} \ \forall f : \mathcal{B} \xrightarrow{1-1} \omega \ \exists S \in \mathcal{S} \ \ G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_{\mathbb{L}}$;
($\mathbb{L}$3) *$\mathcal{S}$ is stable for some forcing adding a dominating real.*

Proof. ($\mathbb{L}$1)⇔($\mathbb{L}$2). Since the Laver forcing has weak fusion ([2], Lemma 2.2.3) and satisfies condition ($*$) ([2], p. 278), we apply Theorem 2.

($\mathbb{L}$1)⇒($\mathbb{L}$3). It is well known that forcing by $\mathbb{L}$ adds dominating reals.

($\mathbb{L}$3)⇒($\mathbb{L}$2). Let $\mathbb{P}$ be a forcing and $V_{\mathbb{P}}[G]$ an extension containing a dominating real and preserving $\mathcal{S}$. By Fact 11, in $V_{\mathbb{P}}[G]$ there is a strongly dominating real. Let $\mathcal{B} \subset {}^{<\omega}\omega$ and $G_\delta(\mathcal{B}) \notin \mathbb{I}_{\mathbb{L}}$. By Fact 12(b), there is a strongly dominating real $z \in G_\delta(\mathcal{B})^{V_{\mathbb{P}}[G]}$ so, by Fact 2, we have $|\widetilde{z} \cap \mathcal{B}| = \aleph_0$. Let $f : \mathcal{B} \to \omega$ be an injection belonging to V. Then clearly $f[\widetilde{z} \cap \mathcal{B}] \in [\omega]^\omega \cap V_{\mathbb{P}}[G]$ so, since $V_{\mathbb{P}}[G] \models$ '$\mathcal{S}$ is a splitting family', there exists $S \in \mathcal{S}$ such that the sets $f[\widetilde{z} \cap \mathcal{B}] \cap S$ and $f[\widetilde{z} \cap \mathcal{B}] \setminus S$ are infinite. Consequently, the sets $\widetilde{z} \cap f^{-1}[S]$ and $\widetilde{z} \cap f^{-1}[\omega \setminus S]$ are infinite too, so, applying Facts 2 and 1 we have $z \in [G_\delta(f^{-1}[S])]^{V_{\mathbb{P}}[G]} \cap [G_\delta(f^{-1}[\omega \setminus S])]^{V_{\mathbb{P}}[G]} = [G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S])]^{V_{\mathbb{P}}[G]}$. According to Fact 12(a), there holds $G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_{\mathbb{L}}$. □

Using the methods described above, it is easy to obtain the following characterization of Cohen-stability of splitting families similar to the characterizations obtained in Theorem 1.

Theorem 7. *If $\mathcal{S}$ is a splitting family on ω, then the f.c.e.:*

($\mathbb{C}$1) *$\mathcal{S}$ is $\mathbb{C}$-stable;*

($\mathbb{C}$2) $\forall f : {}^{<\omega}\omega \xrightarrow{1-1} \omega \ \exists S \in \mathcal{S} \ \ G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_{\mathbb{C}}$.

7. The hierarchy of forcing stability

Theorem 8.

$$\begin{array}{ccccc} \mathbb{L} & \to & \mathbb{M} & \to & \mathbb{S} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{D} & \to & \mathbb{C} & & \mathbb{B} \end{array}$$

where $\mathbb{P} \to \mathbb{Q}$ means: ZFC $\vdash$ "Each $\mathbb{P}$-stable splitting family on ω is $\mathbb{Q}$-stable".

Proof. Since each dominating real is unbounded and each unbounded real is new, using Theorems 4–6 we obtain $\mathbb{L} \to \mathbb{M} \to \mathbb{S}$. Since the Cohen forcing adds an unbounded real, we have $\mathbb{C} \to \mathbb{M}$. Clearly $\mathbb{B}$ adds new reals, so $\mathbb{B} \to \mathbb{S}$, and $\mathbb{D}$ adds a dominating real, thus $\mathbb{D} \to \mathbb{L}$. Clearly, a splitting family is $\mathbb{C}$-stable if it is stable for some forcing adding a Cohen real. So, since the Hechler forcing adds a Cohen real ($x \in 2^\omega$, defined by $x(n) = 0$ iff $f(n)$ is even, where $f \in \omega^\omega$ is a Hechler dominating real) we have $\mathbb{D} \to \mathbb{C}$. $\square$

Proposition 1. *Let $\mathcal{S}$ be a splitting family on ω, $\mathcal{I} \subset P(\omega)$ a tall ideal and $\mathbb{P}$ and $\mathbb{Q}$ forcing notions. For each infinite $I \in \mathcal{I}$ let $\pi_I : \omega \to I$ be a fixed bijection. Then*

(a) *$\mathcal{S}_\mathcal{I} = \bigcup_{I \in \mathcal{I} \cap [\omega]^\omega} \{\pi_I[S] : S \in \mathcal{S}\}$ is a splitting family on ω.*

(b) *If $\mathcal{S}$ remains splitting in an extension $V[G]$, then $\mathcal{S}_\mathcal{I}$ remains splitting in $V[G]$ iff $\mathcal{I}$ remains tall in $V[G]$.*

(c) *If $\mathcal{S}$ is $\mathbb{P}$-stable, then $\mathcal{S}_\mathcal{I}$ is $\mathbb{P}$-stable iff $\mathcal{I}$ is a $\mathbb{P}$-stable tall ideal.*

(d) *If there exists a splitting family which is both $\mathbb{P}$-stable and $\mathbb{Q}$-stable, then $\mathbb{P} \not\to \mathbb{Q}$ for tall ideals implies $\mathbb{P} \not\to \mathbb{Q}$ for splitting families.*

Proof. (a) Clearly, for each $I \in \mathcal{I}$, $\mathcal{S}_I = \{\pi_I[S] : S \in \mathcal{S}\}$ is a splitting family on I. If $B \in [\omega]^\omega$, then, since $\mathcal{I}$ is a tall ideal, there is $I \in \mathcal{I}$ such that $B \cap I$ is an infinite subset of I. So there is $S \in \mathcal{S}$ such that the set $\pi_I[S]$ splits $B \cap I$ and, consequently, B.

(b) Let $\mathcal{S}$ be a splitting family in $V[G]$. If $\mathcal{S}_\mathcal{I}$ is splitting in $V[G]$ and $X \in [\omega]^\omega \cap V[G]$, then for some $I \in \mathcal{I} \cap [\omega]^\omega$ and some $S \in \mathcal{S}$ we have $|\pi_I[S] \cap X| = \aleph_0$, which implies $|I \cap X| = \aleph_0$. Thus $\mathcal{I}$ is a tall ideal in $V[G]$. Conversely, if $\mathcal{I}$ is a tall ideal in $V[G]$, then for each $X \in [\omega]^\omega \cap V[G]$ there is $I \in \mathcal{I}$ such that $|I \cap X| = \aleph_0$. Since, clearly $\{\pi_I[S] : S \in \mathcal{S}\}$ is a splitting family on I in $V[G]$, there is $S \in \mathcal{S}$ such that the set $\pi_I[S]$ splits $I \cap X$ and, consequently, X.

(c) and (d) follow from (b). $\square$

Similarly, there holds.

Proposition 2. *Let $\mathcal{S}$ be a splitting family on ω, $\mathcal{A} \subset [\omega]^\omega$ a mad family and $\mathbb{P}$ and $\mathbb{Q}$ forcing notions. For each $A \in \mathcal{A}$ let $\pi_A : \omega \to A$ be a fixed bijection. Then*

(a) *$\mathcal{S}_A = \bigcup_{A \in \mathcal{A}} \{\pi_A[S] : S \in \mathcal{S}\}$ is a splitting family on ω;*

(b) *If $\mathcal{S}$ remains splitting in an extension $V[G]$, then $\mathcal{S}_A$ is splitting in $V[G]$ iff $\mathcal{A}$ remains a mad family in $V[G]$;*

(c) *If $\mathcal{S}$ is $\mathbb{P}$-stable, then $\mathcal{S}_\mathcal{A}$ is $\mathbb{P}$-stable iff $\mathcal{A}$ is a $\mathbb{P}$-stable mad family;*

(d) *If there exists a splitting family which is both $\mathbb{P}$-stable and $\mathbb{Q}$-stable, then $\mathbb{P} \not\to \mathbb{Q}$ for mad families implies $\mathbb{P} \not\to \mathbb{Q}$ for splitting families.*

Theorem 9. *Let $\mathbb{P}, \mathbb{Q} \in \{\mathbb{S}, \mathbb{M}, \mathbb{L}, \mathbb{C}, \mathbb{B}, \mathbb{D}\}$. Then $\mathbb{P} \not\to \mathbb{Q}$ for tall ideals (or for mad families) implies $\mathbb{P} \not\to \mathbb{Q}$ for splitting families.*

Proof. By [6] (see also [1] p. 176) if $\mathbb{P}$ is a Suslin forcing, then $[\omega]^\omega$ is a $\mathbb{P}$-stable splitting family. $\mathbb{B}$ and $\mathbb{D}$ are Suslin forcings (see [1], p. 168) so the splitting family $[\omega]^\omega$ is $\mathbb{B}$ and $\mathbb{D}$-stable. According to Theorem 8 it is $\mathbb{S}$, $\mathbb{M}$, $\mathbb{L}$ and $\mathbb{C}$-stable too. Now we can apply (d) of Proposition 1, or (d) of Proposition 2. $\square$

Theorem 10. *In the diagram given in Theorem 8 there are no additional implications provable in ZFC.*

Proof. In the sequel we use the previous theorem. According to Theorem 3.3.2 of [2], under CH there is a $\mathbb{C}$-stable mad family which is not $\mathbb{B}$-stable. So for splitting families we have $\mathbb{C} \not\to \mathbb{B}$, and according to Theorem 8, $\mathbb{M} \not\to \mathbb{B}$ and $\mathbb{S} \not\to \mathbb{B}$. By Theorem 3.6.1 of [2], $\mathrm{add}(\mathcal{N}) = \mathfrak{c}$ implies there is a $\mathbb{B}$-stable mad family which is not $\mathbb{M}$-stable. So, for splitting families we have $\mathbb{B} \not\to \mathbb{M}$, and, consequently, $\mathbb{B} \not\to \mathbb{C}, \mathbb{L}, \mathbb{D}$ and $\mathbb{S} \not\to \mathbb{M}, \mathbb{C}$.

Since the forcings $\mathbb{L}$ and $\mathbb{D}$ produce dominating reals, they kill all mad families. Consequently, $\mathbb{C}, \mathbb{M}, \mathbb{S} \not\to \mathbb{L}, \mathbb{D}$.

According to Theorem 3.7.3 of [2] there is a $\mathbb{L}$-stable tall ideal which is not $\mathbb{C}$-stable. Thus for splitting families we have $\mathbb{L} \not\to \mathbb{C}$, which implies $\mathbb{L} \not\to \mathbb{D}$ and $\mathbb{M} \not\to \mathbb{C}$.

Finally, by Theorem 3.7.4 of [2] there is a $\mathbb{D}$-stable, $\mathbb{B}$-unstable tall ideal. Thus for splitting families we have $\mathbb{D} \not\to \mathbb{B}$ and, consequently, $\mathbb{L} \not\to \mathbb{B}$. $\square$

8. On the existence of stable and unstable splitting families

Theorem 11. *Some splitting families are killed by all forcings which add new reals. Some forcings kill all splitting families on ω. Consequently, each splitting family is killed by some forcing.*

Proof. According to Ramsey's Theorem, the family $\mathcal{S}$ of all infinite chains and antichains of the tree ${}^{<\omega}2$ is a splitting family (on the set ${}^{<\omega}2$). If $V[G]$ is a generic extension containing a new real, then in $V[G]$ the tree ${}^{<\omega}2$ obtains a new branch which is almost disjoint with each element of $\mathcal{S}$, so $\mathcal{S}$ is not a splitting family in $V[G]$. On the other hand, if a forcing $\mathbb{P}$ collapses $\mathfrak{c}$ to $\aleph_0$, then each splitting family becomes countable, and, consequently, it is not a splitting family any more. Referee's remark: A more interesting forcing which kills all splitting families is the Mathias forcing. $\square$

Thus the question concerning the existence of unstable splitting families is settled. What is going on with stability? Firstly by [6] (see also [1], p. 176) there holds.

Fact 13. *If $\mathbb{P}$ is a Suslin forcing, then $[\omega]^\omega$ is a $\mathbb{P}$-stable splitting family on ω.*

So, since $\mathbb{B}$ and $\mathbb{D}$ are Suslin forcings (see e.g. [1], p. 168) according to Theorem 8 we have

Corollary 3. *For $\mathbb{P} \in \{\mathbb{S}, \mathbb{M}, \mathbb{L}, \mathbb{C}, \mathbb{B}, \mathbb{D}\}$, $[\omega]^\omega$ is a $\mathbb{P}$-stable splitting family.*

Now, are there stable splitting families which are smaller than $[\omega]^\omega$? Concerning Cohen forcing, by Theorem 10 of [8] there holds:

Theorem 12. *Let $\mathcal{S}$ be a splitting family on ω. If $|\mathcal{S}| < \mathrm{cov}(\mathcal{M})$, then $\mathcal{S}$ is $\mathbb{C}$-stable.*

Regarding the Cohen stability we remark that, according to Theorem 9 of [8], $\aleph_0$-splitting families are $\mathbb{C}$-stable and, conversely, each $\mathbb{C}$-stable splitting family has a restriction to some $B \in [\omega]^\omega$ which is an $\aleph_0$-splitting family. ($\aleph_0$-splitting families are introduced by Malyhin in [10].) In the sequel, using the methods from [2], we generalize Theorem 12.

Lemma 6. *Let $\mathcal{T} = {}^{<\omega}2$ (respectively $\mathcal{T} = {}^{<\omega}\omega$) and $\mathbb{R} = 2^\omega$ (respectively $\mathbb{R} = \omega^\omega$). Then for each family $\mathcal{S} \subset [\omega]^\omega$ the following conditions are equivalent*

(a) *$\mathcal{S}$ is a splitting family;*

(b) $\forall f : \mathcal{T} \xrightarrow{1-1} \omega \;\; \mathbb{R} = \bigcup_{S \in \mathcal{S}} G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S])$;

(c) $\forall \mathcal{B} \subset \mathcal{T} \;\; \forall f : \mathcal{B} \xrightarrow{1-1} \omega \;\; G_\delta(\mathcal{B}) \subset \bigcup_{S \in \mathcal{S}} G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S])$.

The statement remains true if in (b) *and* (c) *we replace '$1-1$' by 'finite-to-one'.*

Proof. (a) $\Rightarrow$ (c). Let condition (a) hold, let $\mathcal{B} \subset \mathcal{T}$, $f : \mathcal{B} \xrightarrow{1-1} \omega$ and $x \in G_\delta(\mathcal{B})$. Then the set $f[\widetilde{x} \cap \mathcal{B}] \subset \omega$ is infinite, so there is $S \in \mathcal{S}$ such that $f[\widetilde{x} \cap \mathcal{B}] \cap S = \{f(x \upharpoonright n) : n \in N\}$ and $f[\widetilde{x} \cap \mathcal{B}] \setminus S = \{f(x \upharpoonright n) : n \in M\}$, where $N, M \in [\omega]^\omega$. Hence $\{x \upharpoonright n : n \in N\} \subset \widetilde{x} \cap f^{-1}[S]$ so $x \in G_\delta(f^{-1}[S])$ and similarly $x \in G_\delta(f^{-1}[\omega \setminus S])$.

(c) $\Rightarrow$ (b) follows from the equality $G_\delta(\mathcal{T}) = \mathbb{R}$.

(b) $\Rightarrow$ (a). Let condition (b) hold. Let $A \in [\omega]^\omega$. W.l.o.g. suppose $|\omega \setminus A| = \aleph_0$. We choose an $x \in \mathbb{R}$ and a bijection $f : \mathcal{T} \to \omega$ such that $f[\widetilde{x}] = A$. By (b) there exists $S \in \mathcal{S}$ such that the sets $\widetilde{x} \cap f^{-1}[S]$ and $\widetilde{x} \cap f^{-1}[\omega \setminus S]$ are infinite, thus, since f is an bijection, the sets $A \cap S$ and $A \setminus S$ are infinite too, that is the set S splits the set A. $\square$

Theorem 13. *Let* $\mathbb{P} = \mathrm{Borel}(\mathbb{R}_{\mathbb{P}}) \setminus \mathbb{I}_{\mathbb{P}}$ *be a real forcing such that*
(i) $\forall p \in \mathbb{P}\ \ \mathrm{cov}(\mathbb{I}_{\mathbb{P}} \upharpoonright p) = \mathrm{cov}(\mathbb{I}_{\mathbb{P}})$*;*
(ii) *A splitting family* $\mathcal{S}$ *on* ω *is* $\mathbb{P}$*-stable iff*
$\forall \mathcal{B} \in \mathcal{T}_{\mathbb{P}}^{+}\ \ \forall f : \mathcal{B} \xrightarrow{1-1} \omega\ \ \exists S \in \mathcal{S}\ \ G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_{\mathbb{P}}$.
Then each splitting family $\mathcal{S}$ *of size* $< \mathrm{cov}(\mathbb{I}_{\mathbb{P}})$ *is* $\mathbb{P}$*-stable.*
The statement remains true if in (ii) *we replace* '$1-1$' *by 'finite-to-one'.*

Proof. Let $\mathcal{S}$ be a splitting family and $|\mathcal{S}| < \mathrm{cov}(\mathbb{I}_{\mathbb{P}})$. Let $\mathcal{B} \in \mathcal{T}_{\mathbb{P}}^{+}$ and $f : \mathcal{B} \xrightarrow{1-1} \omega$. By the previous lemma we have $G_\delta(\mathcal{B}) \subset \bigcup_{S \in \mathcal{S}} G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S])$, so, since $p = G_\delta(\mathcal{B}) \notin \mathbb{I}_{\mathbb{P}}$ and since by (i) $\mathrm{cov}(\mathbb{I}_{\mathbb{P}} \upharpoonright p) = \mathrm{cov}(\mathbb{I}_{\mathbb{P}}) > |\mathcal{S}|$, there must be an $S \in \mathcal{S}$ satisfying $G_\delta(f^{-1}[S]) \cap G_\delta(f^{-1}[\omega \setminus S]) \notin \mathbb{I}_{\mathbb{P}}$. By (ii) $\mathcal{S}$ is a $\mathbb{P}$-stable splitting family. □

Theorem 14. *Let* $\mathcal{S}$ *be a splitting family on* ω*. Then*
(a) $|\mathcal{S}| < \mathfrak{c} \Rightarrow \mathcal{S}$ *is* $\mathbb{S}$*-stable.*
(b) $|\mathcal{S}| < \mathfrak{d} \Rightarrow \mathcal{S}$ *is* $\mathbb{M}$*-stable.*
(c) $|\mathcal{S}| < \mathfrak{b} \Rightarrow \mathcal{S}$ *is* $\mathbb{L}$*-stable.*
(d) $|\mathcal{S}| < \mathrm{cov}(\mathcal{M}) \Rightarrow \mathcal{S}$ *is* $\mathbb{C}$*-stable.*
(e) $|\mathcal{S}| < \mathrm{cov}(\mathcal{N}) \Rightarrow \mathcal{S}$ *is* $\mathbb{B}$*-stable.*

Proof. A σ-ideal $\mathbb{I}_{\mathbb{P}}$ is called homogeneous iff for each Borel set $B \notin \mathbb{I}_{\mathbb{P}}$ there exists a function $f : \mathbb{R}_{\mathbb{P}} \to B$ such that $f^{-1}[I] \in \mathbb{I}_{\mathbb{P}}$ for every $I \in \mathbb{I}_{\mathbb{P}}$. Then (see [12], p. 15) for each $B \in \mathrm{Borel}(\mathbb{R}_{\mathbb{P}}) \setminus \mathbb{I}_{\mathbb{P}}$ there holds $\mathrm{cov}(\mathbb{I}_{\mathbb{P}} \upharpoonright B) = \mathrm{cov}(\mathbb{I}_{\mathbb{P}})$. Thus, since the ideals corresponding to the forcings $\mathbb{S}, \mathbb{M}, \mathbb{L}, \mathbb{C}$ and $\mathbb{B}$ are homogeneous (see [12]) these forcings satisfy condition (i) of Theorem 13. By Corollaries 1 and 2 they satisfy condition (ii) of Theorem 13 as well. Finally, $\mathrm{cov}(\mathbb{I}_{\mathbb{S}}) = \mathfrak{c}$, $\mathrm{cov}(\mathbb{I}_{\mathbb{M}}) = \mathfrak{d}$ (see [1]) and $\mathrm{cov}(\mathbb{I}_{\mathbb{L}}) = \mathfrak{b}$ (see [13]) and we apply Theorem 13. □

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